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EMPIRICAL BAYES TWO-TAIL TESTS
IN A DISCRETE EXPONENTIAL FAMILY*

by

TaChen Liang
Wayne State University

Technical Report #96-17

Department of Statistics
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May 1996

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Abstract

This paper deals with the problem of testing $H_0 : \theta \in [\theta_1, \theta_2]$ versus $H_1 : \theta \notin [\theta_1, \theta_2]$, where $0 < \theta_1 < \theta_2 < \infty$, for the parameter θ in a discrete exponential family via the empirical Bayes approach. First, the behavior of the Bayes test is examined. Then the empirical Bayes test is constructed by mimicking the behavior of the Bayes test. The asymptotic optimality of the empirical Bayes tests is investigated. It is shown that, under very mild regularity conditions, the proposed empirical Bayes test is asymptotically optimal and its associated Bayes risk converges to the minimum Bayes risk with a rate of convergence of order $O(\exp(-\tau n))$ for some $\tau > 0$, where n is the number of historical data at hand for the present testing problem.

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1. Introduction

Since Robbins (1956, 1964), the empirical Bayes approach to statistical problems has generated considerable interest among the researchers. Empirical Bayes procedures have been extensively studied in the literature. To name a few, Lin (1975) and Singh (1976, 1979) considered empirical Bayes estimation for the one parameter exponential family, and Singh and Wei (1992) studied empirical Bayes estimation in a nonnegative exponential family. Johns and Van Ryzin (1972), Van Houwelingen (1976) and Stijnen (1985) studied empirical Bayes tests for one-parameter continuous exponential family, while Johns and Van Ryzin (1971) and Liang (1988) considered empirical Bayes one-sided tests for a discrete exponential family. Empirical Bayes procedures have also been studied for non-exponential family distributions, see Huang (1995) and the cited references there. In this paper, we study the empirical Bayes two-tail tests in a discrete exponential family. Our research interest is motivated by Wei (1991) in which an empirical Bayes two tail test was investigated.

Let X denote a random variable arising from a discrete exponential family with probability function

$$f(x|\theta) = a(x)\beta(\theta)\theta^x, \quad x = 0, 1, 2, \ldots; \ 0 < \theta < Q \tag{1.1}$$

where $a(x) > 0$ for all $x = 0, 1, 2, \ldots$, and Q may be finite or infinite. Consider the problem of testing $H_0 : \theta \in [\theta_1, \theta_2]$ versus $H_1 : \theta \notin [\theta_1, \theta_2]$, where $0 < \theta_1 < \theta_2 < \infty$ are known constants. This type of testing hypotheses may arise from social studies, engineering problems or environmental sciences. For example, one may like to see if there is any change of the frequency of rainfalls during the last several years. Let i, $i = 0, 1$, denote an action deciding in favor of the hypothesis H_i. For the parameter θ and action i, the loss function is defined to be:

$$L(\theta, i) = (1-i)(\theta-\theta_1)(\theta-\theta_2)I(\theta \notin [\theta_1, \theta_2]) + i(\theta-\theta_1)(\theta_2-\theta)I(\theta \in [\theta_1, \theta_2]), \tag{1.2}$$

where $I(A)$ denotes the indicator function of the event A.

If we let $\theta_0 = (\theta_1 + \theta_2)/2$, $c = (\theta_2 - \theta_1)/2$, then the testing hypotheses can be written as $H_0 : |\theta - \theta_0| \leq c$ versus $H_1 : |\theta - \theta_0| > c$, and the loss function is

$$L(\theta, i) = (1-i)[(\theta-\theta_0)^2 - c^2]I(|\theta-\theta_0| > c) + i[c^2 - (\theta-\theta_0)^2]I(|\theta-\theta_0| \leq c). \tag{1.2$'$}$$

Note that $0 < c < \theta_0$.

It is assumed that the parameter θ is a realization of a random variable Θ having an unknown prior distribution G over the parameter space $\Omega = (0, Q)$.

Let $\mathcal{X}$ be the sample space generated by X. A test δ is defined to be a mapping from $\mathcal{X}$ into $[0,1]$, so that $\delta(x)$ is the probability of taking action 0 when $X = x$ is observed. That is, $\delta(x) = P\{\text{accept } H_0 | X = x\}$. Then for each $x = 0, 1, 2, \ldots$

$$L(\theta, \delta(x)) = \delta(x)[(\theta - \theta_0)^2 - c^2] - [(\theta - \theta_0)^2 - c^2] I(|\theta - \theta_0| \le c).$$

We consider only those priors G such that $\int \theta^2 dG(\theta) < \infty$ to insure that the Bayes risk is always finite, and hence, the testing problem is meaningful. This assumption always holds when Q is finite.

Let $r(G, \delta)$ denote the Bayes risk of the test δ. By Fubini's theorem, we have:

$$r(G, \delta) = \sum_{x=0}^{\infty} \delta(x) H(x) f(x) + C, \tag{1.3}$$

where

$$H(x) = \varphi(x+1)\varphi(x) - 2\theta_0 \varphi(x) + \theta_0^2 - c^2; \tag{1.4}$$

$\varphi(x) = E[\Theta | X = x] = \frac{h(x+1)}{h(x)}$ is the posterior mean of Θ given $X = x$; $f(x) = \int f(x|\theta)$ $dG(\theta) = a(x) \int \beta(\theta) \theta^x dG(\theta) = a(x) h(x)$ is the marginal probability function of X, and $C = \int_{\Omega} [c^2 - (\theta - \theta_0)^2] I(|\theta - \theta_0| \le c) dG(\theta)$.

From (1.3), a Bayes test, say δ_G, is clearly given by:

$$\delta_G(x) = \begin{cases} 1 & \text{if } H(x) < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.5}$$

The minimum Bayes risk is:

$$r(G, \delta_G) = \sum_{x=0}^{\infty} \delta_G(x) H(x) f(x) + C. \tag{1.6}$$

When the prior distribution G is unknown, this testing problem has been studied by Wei (1991) via the empirical Bayes approach. Our interest is motivated by the result of

Wei (1991), in which under certain strong conditions, Wei (1991) proved that his proposed empirical Bayes test may achieve a rate of convergence with order near the "best" possible rate $O(n^{-1})$, where n is the number of historical data at hand for the present testing problem. Basically, Wei's approach is along the line of Johns and Van Ryzin (1971). Though this rate sounds good, the conditions can be reduced and the rate can be much improved.

This paper is organized in the following way. In Section 2, we first examine the behavior of the Bayes test δ_G, and then construct an empirical Bayes test that $\tilde{\delta}_n$ by mimicking the behavior of the Bayes test δ_G. The asymptotic optimality of the empirical Bayes test $\tilde{\delta}_n$ is investigated in Section 3. It is shown that under very mild regularity conditions, the empirical Bayes test $\tilde{\delta}_n$ is asymptotically optimal of order $O(\exp(-cn))$ for some constant $c > 0$. This result much improves that of Wei (1991). Finally, certain examples and some further results are given in Section 4.

2. Construction of Empirical Bayes Test

2.1 Properties of Bayes test δ_G

First note that the posterior mean $\varphi(x) = \frac{h(x+1)}{h(x)}$ is increasing in x for $x = 0, 1, 2, \ldots$; and is strictly increasing if the prior distribution G is non-degenerate. Let

$$A = \{x \mid \varphi(x) \geq \theta_0 + c\}, \text{ and}$$

$$B = \{x \mid \varphi(x) < \theta_0 - c\}.$$

Define

$$a^* = \begin{cases} \inf A & \text{if } A \neq \phi, \\ \infty & \text{if } A = \phi; \end{cases}$$

$$b^* = \begin{cases} \sup B & \text{if } B \neq \phi \\ -1 & \text{if } B = \phi; \end{cases} \tag{2.1}$$

where ϕ denotes the empty set. By the definition of a^* and b^*, and by the increasing property of $\varphi(x)$, we have: $b^* < a^*$ and

$$\begin{aligned} &\varphi(x) \geq \varphi(a^*) \geq \theta_0 + c \quad \text{for all} \quad x \geq a^*, \\ &\varphi(y) \leq \varphi(b^*) < \theta_0 - c \quad \text{for all} \quad y \leq b^*. \end{aligned} \tag{2.2}$$

We consider only non-degenerate prior distributions G so that $\varphi(x)$ is strictly increasing in x.

Proposition 2.1

(a) For $x \geq a^*, H(x) > 0$.

(b) For $x \leq b^*, H(x) > 0$.

Proof: (a) By the strictly increasing property of $\varphi(x)$ and (2.2), for $x \geq a^*$,

$$
\begin{aligned}
H(x) &= \varphi(x+1)\varphi(x) - 2\theta_0\varphi(x) + \theta_0^2 - c^2 \\
&> \varphi^2(x) - 2\theta_0\varphi(x) + \theta_0^2 - c^2 \\
&= (\varphi(x) - \theta_0)^2 - c^2 \\
&\geq 0.
\end{aligned}
$$

Part (b) can be obtained in a similar way. The detail is omitted. □

Let $D = \{x | H(x) < 0\}$. If $D \neq \phi$, define $d_2 = \sup D$ and $d_1 = \inf D$. From Proposition 2.1, $b^* < d_1 \leq d_2 < a^*$.

Proposition 2.2 If $\varphi(x)$ is linear in x, the set D consists of all integers between (including) d_1 and d_2.

Proof: Under the assumption, $\varphi(x) = ax + b$ for some constants $a > 0$ and b. Plugging this linear form of $\varphi(x)$ into $H(x)$, we obtain

$$H(x) = a^2 \; x^2 + 2x[ab - a\theta_0 + a^2] + (b - \theta_0)^2 + ab - c^2,$$

which is a convex function of x. Therefore, Proposition 2.2 is concluded by noting the definitions of d_1 and d_2. □

Proposition 2.2 well describes the behavior of the Bayes test δ_G. That is, as soon as d_1 and d_2 are determined, then the Bayes test δ_G is determined. However, without the assumption of the linearity of the posterior mean $\varphi(x)$, it is not known whether Proposition

2.2 still holds or not. Even though, from Proposition 2.1, the Bayes test δ_G can be represented as:

$$\delta_G(x) = \begin{cases} 0 & \text{if } (x \leq b^*) \text{ or } (x \geq a^*) \\ & \text{or } (b^* < x < a^* \text{ and } H(x) \geq 0); \\ \\ 1 & \text{if } b^* < x < a^* \text{ and } H(x) < 0. \end{cases} \tag{2.3}$$

It should be noted that Proposition 2.1 is based on the increasing property of the posterior mean $\varphi(x)$. Hence, in the following, to construct an empirical Bayes test, we first construct a monotone empirical Bayes estimator for $\varphi(x)$.

2.2. Empirical Bayes Framework and Estimation of $\varphi(x)$

In the empirical Bayes approach, let $(X_i, \Theta_i), i = 1, 2, \ldots$, be iid with (X, Θ), where $X_i, i = 1, 2, \ldots$, are observable and $\Theta_i, i = 1, 2, \ldots$ are not observable. At time $n+1$, $\underset{\sim}{X}(n) = (X_1, \ldots, X_n)$ denotes the historical data and X_{n+1} denotes the present random observation and one is interested in testing $H_{0,n+1}: \quad |\theta_{n+1} - \theta_0| \leq c$ versus $H_{1,n+1}$: $|\theta_{n+1} - \theta_0| > c$ with the loss $L(\theta_{n+1}, i)$ given in (1.2′), where θ_{n+1} is a realization of the random parameter Θ_{n+1}. A test δ_n, called as an empirical Bayes test, is a function of the present observation $X_{n+1} = x$ and the historical data $\underset{\sim}{X}(n)$, such that $\delta_n(x, \underset{\sim}{X}(n)) \equiv \delta_n(x)$ is the probability of accepting the hypothesis $H_{0,n+1}$. Let $r(G, \delta_n | \underset{\sim}{X}(n))$ be the Bayes risk of the empirical Bayes test δ_n conditioning on $\underset{\sim}{X}(n)$. Also, let $r(G, \delta_n) = E_{\underset{\sim}{X}(n)}$ $r(G, \delta_n | \underset{\sim}{X}(n))$ denote the overall Bayes risk of δ_n, where the expectation $E_{\underset{\sim}{X}(n)}$ is taken with respect to the probability measure generated by $\underset{\sim}{X}(n)$.

Since $r(G, \delta_G)$ is the minimum Bayes risk, $r(G, \delta_n) \geq r(G, \delta_G)$ for all n. The non-negative regret Bayes risk $r(G, \delta_n) - r(G, \delta_G)$ is used as a measure of performance of the empirical Bayes test δ_n. A sequence of empirical Bayes tests $\{\delta_n\}_{n=1}^{\infty}$ is said to be asymptotically optimal relative to the prior distribution G if $r(G, \delta_n) - r(G, \delta_G) = o(1)$; $\{\delta_n\}_{n=1}^{\infty}$ is said to be asymptotically optimal relative to the prior distribution G of order $\{\alpha_n\}$ if $r(G, \delta_n) - r(G, \delta_G) = O(\alpha_n)$, where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that $\lim_{n \to \infty} \alpha_n = 0$.

Wei (1991) had studied an empirical Bayes two-tail test for this testing problem. Under certain strong regularity conditions, Wei (1991) proved that his proposed empirical

Bayes test is asymptotically optimal of order near $O(n^{-1})$. His empirical Bayes test is based on an alternative form of the Bayes test that

$$\delta_G(x) = 1 \quad \text{iff} \quad \gamma(x) \equiv h(x)H(x) = h(x+2) - 2\theta_0 h(x+1) + h(x)[\theta_0^2 - c^2] < 0.$$

He then constructed an empirical Bayes estimator $\gamma_n(x)$ for $\gamma(x)$, and his empirical Bayes test δ_n^W is defined as: $\delta_n^W(x) = 1$ iff $\gamma_n(x) < 0$. Though δ_n^W is asymptotically optimal, such an approach ignores the monotinicity of the posterior mean $\varphi(x)$. In the following, we first construct a monotone empirical Bayes estimator, say $\tilde{\varphi}_n(x)$, for $\varphi(x)$. Then based on $\tilde{\varphi}_n(x)$, we propose an empirical Bayes test $\tilde{\delta}_n$ which possesses properties similar to that of Proposition 2.1.

We let $\{w(x)\}_{x=0}^{\infty}$ be a sequence of positive numbers such that the following condition holds.

Condition C1. $\sum_{x=0}^{\infty} w(x) < \infty$ and both the sequences $\left\{\frac{w(x)}{a(x)}\right\}_{x=0}^{\infty}$ and $\left\{\frac{w(x)}{a(x+1)}\right\}_{x=0}^{\infty}$ are nonincreasing in x, and bounded above by 1.

Based on the historical data, let $m_n = \min(X_1, \ldots, X_n)$ and $M_n = \max(X_1, \ldots, X_n) - 1$. For each $x = 0, 1, \ldots,$ define

$$\begin{cases} f_n(x) = \frac{1}{n} \sum_{j=1}^{n} I_{\{x\}}(X_j), \\ h_n(x) = \frac{f_n(x)}{a(x)} + \epsilon_n; \end{cases} \tag{2.4}$$

where $\epsilon_n > 0$ is such that $\epsilon_n = o(n^{-1})$. For each $y = 0, 1, \ldots,$ let

$$\begin{cases} \psi_n(y) = \sum_{x=0}^{y} h_n(x+1)w(x), \\ \psi(y) = \sum_{x=0}^{y} h(x+1)w(x), \\ K_n(y) = \sum_{x=0}^{y} h_n(x)\, w(x), \\ K(y) = \sum_{x=0}^{y} h(x)\, w(x). \end{cases} \tag{2.5}$$

Also, let $\psi_n(-1) = \psi(-1) = K_n(-1) = K(-1) = 0$. Next, define

$$\tilde{\varphi}_n(m_n) = \min_{m_n \le y \le M_n} \left[\frac{\psi_n(y)}{K_n(y)} \right], \tag{2.6}$$

and for each $x = m_n + 1, \ldots, M_n$, recursively define

$$\tilde{\varphi}_n(x) = \min_{x \le y \le M_n} \left[\frac{\psi_n(y) - \tilde{\psi}_n(x-1)}{K_n(y) - K_n(x-1)} \right], \tag{2.7}$$

where

$$\tilde{\psi}_n(x) = \sum_{i=0}^{x} \varphi_n^*(i)\ h_n(i)\ w(i) \tag{2.8}$$

and $\varphi_n^*(i) = \tilde{\varphi}_n(i)$ if $m_n \le i \le x$ and $\varphi_n^*(i) = \frac{h_n(i+1)}{h_n(i)}$ for $i < m_n$. Finally, for $0 \le x < m_n$, define $\tilde{\varphi}_n(x) = \tilde{\varphi}_n(m_n)$ and for $x > M_n$ let $\tilde{\varphi}_n(x) = \tilde{\varphi}_n(M_n)$.

It should be noted that $\{\tilde{\varphi}_n(x)\}_{x=m_n}^{M_n}$ is the isotonic regression of $\{\varphi_n(x)\}_{x=m_n}^{M_n}$ with weights $\{h_n(x)w(x)\}_{x=m_n}^{M_n}$ where $\varphi_n(x) = \frac{h_n(x+1)}{h_n(x)}$ which is a consistent estimator of $\varphi(x)$. Therefore, $\tilde{\varphi}_n(x)$ is also a consistent estimator of $\varphi(x)$, see BBBB (1972). We state the result as a Lemma without providing the proof.

Lemma 2.1 For each $x = 0, 1, \ldots, \tilde{\varphi}_n(x)$ converges to $\varphi(x)$ in probability.

2.3 Empirical Bayes Test $\tilde{\delta}_n$

By mimicking the form of (1.4) - (1.5), an empirical Bayes test $\tilde{\delta}_n$ is constructed as follows: For each $X_{n+1} = x$, let

$$\tilde{H}_n(x) = \tilde{\varphi}_n(x+1)\tilde{\varphi}_n(x) - 2\theta_0\tilde{\varphi}_n(x) + \theta_0^2 - c^2, \tag{2.9}$$

and

$$\tilde{\delta}_n(x) = \begin{cases} 1 & \text{if } \tilde{H}_n(x) < 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

Given $\underset{\sim}{X}(n)$, the conditional Bayes risk of $\tilde{\delta}_n$ is

$$r(G, \tilde{\delta}_n | \underset{\sim}{X}(n)) = \sum_{x=0}^{\infty} \tilde{\delta}_n(x) H(x) f(x) + C, \tag{2.11}$$

and the overall Bayes risk of $\tilde{\delta}_n$ is

$$r(G, \tilde{\delta}_n) = \sum_{x=0}^{\infty} E_{\underset{\sim}{X}(n)} [\tilde{\delta}_n(x)] H(x) f(x) + C. \tag{2.12}$$

For each n, let $A_n = \{x | \tilde{\varphi}_n(x) \geq \theta_0 + c\}$ and $B_n = \{x | \tilde{\varphi}_n(x) < \theta_0 - c\}$. Define

$$\tilde{a}_n = \begin{cases} \inf A_n & \text{if } A_n \neq \phi, \\ \infty & \text{if } A_n = \phi; \end{cases}$$

$$\tilde{b}_n = \begin{cases} \sup B_n & \text{if } B_n \neq \phi, \\ -1 & \text{if } B_n = \phi. \end{cases} \tag{2.13}$$

Note that $\tilde{\varphi}_n(x)$ is nondecreasing in x. Therefore, by the definitions of $\tilde{a}_n$ and $\tilde{b}_n$, $\tilde{b}_n < \tilde{a}_n$, and

$$\begin{cases} \tilde{\varphi}_n(x) \geq \tilde{\varphi}_n(\tilde{a}_n) \geq \theta_0 + c & \text{for all } x > \tilde{a}_n, \\ \tilde{\varphi}_n(y) \leq \tilde{\varphi}_n(\tilde{b}_n) < \theta_0 - c & \text{for all } y < \tilde{b}_n. \end{cases} \tag{2.14}$$

By the nondecreasing property of $\tilde{\varphi}_n(x)$ and (2.14), similar to Proposition (2.1), $\tilde{H}_n(x) \geq 0$ for all $x \geq \tilde{a}_n$ and for all $x \leq \tilde{b}_n$. Hence, the empirical Bayes test $\tilde{\delta}_n$ can be represented as:

$$\tilde{\delta}_n(x) = \begin{cases} 0 & \text{if } (x \leq \tilde{b}_n) \text{ or } (x \geq \tilde{a}_n) \\ & \text{or } (\tilde{b}_n < x < \tilde{a}_n \text{ and } \tilde{H}_n(x) \geq 0), \\ 1 & \text{if } \tilde{b}_n < x < \tilde{a}_n \text{ and } \tilde{H}_n(x) < 0 \end{cases} \tag{2.15}$$

which is similar to that of (2.3).

Remark 2.1

(a) Since $\{\tilde{\varphi}_n(x)\}_{x=m_n}^{M_n}$ is the isotonic regression of $\{\varphi_n(x)\}_{x=m_n}^{M_n}$ with weights $\{h_n(x)w(x)\}_{x=m_n}^{M_n}$, by BBBB (1972) and by the definitions of $\tilde{\psi}_n(z)$ and $\psi_n(z)$, $\tilde{\psi}_n(z) \leq \psi_n(z)$ for all $z = 0, 1, \ldots, M_n$. Therefore, for each $x = m_n, \ldots, M_n$,

$$\begin{aligned} \tilde{\varphi}_n(x) &= \min_{x \leq y \leq M_n} \left[\frac{\psi_n(y) - \tilde{\psi}_n(x-1)}{K_n(y) - K_n(x-1)} \right] \\ &\geq \min_{x \leq y \leq M_n} \left[\frac{\psi_n(y) - \psi_n(x-1)}{K_n(y) - K_n(x-1)} \right]. \end{aligned} \tag{2.16}$$

(b) Following Puri and Singh (1990), the isotonic regression estimators $\{\tilde{\varphi}_n(x)\}_{x=m_n}^{M_n}$ can also be derived in an alternative way.

Define

$$\tilde{\tilde{\varphi}}_n(M_n) = \max_{m_n \le y \le M_n} \left[\frac{\psi_n(M_n) - \psi_n(y-1)}{K_n(M_n) - K_n(y-1)} \right], \tag{2.17}$$

and for each $x = M_n - 1, \ldots, m_n$, recursively define

$$\tilde{\tilde{\varphi}}_n(x) = \max_{m_n \le y \le x} \left[\frac{[\psi_n(M_n) - \psi_n(y-1)] - \sum_{i=x+1}^{M_n} \tilde{\tilde{\varphi}}_n(i) h_n(i) w(i)}{K_n(x) - K_n(y-1)} \right] . \tag{2.18}$$

Then, $\tilde{\tilde{\varphi}}_n(x) = \tilde{\varphi}_n(x)$ for all $x = m_n, \ldots, M_n$. Also,

$$\sum_{i=x+1}^{M_n} \tilde{\tilde{\varphi}}_n(i) h_n(i) w(i) \ge \sum_{i=x+1}^{M_n} h_n(i+1) w(i) = \psi_n(M_n) - \psi_n(x). \tag{2.19}$$

Combining (2.18) and (2.19) yields

$$\begin{aligned} \tilde{\varphi}_n(x) &\le \max_{m_n \le y \le x} \left[\frac{[\psi_n(M_n) - \psi_n(y-1)] - [\psi_n(M_n) - \psi_n(x)]}{K_n(x) - K_n(y-1)} \right] \\ &= \max_{m_n \le y \le x} \left[\frac{\psi_n(x) - \psi_n(y-1)}{K_n(x) - K_n(y-1)} \right] . \end{aligned} \tag{2.20}$$

3. The Main Results

We state our main results of the paper as two theorems as follows.

Theorem 3.1 Suppose $\int \theta^2 \, dG(\theta) < \infty$ and Condition C1 holds. Then, the empirical Bayes test $\tilde{\delta}_n$ is asymptotically optimal in the sense that $r(G, \tilde{\delta}_n) - r(G, \delta_G) = o(1)$.

Theorem 3.2 Suppose that $\int \theta^2 \, dG(\theta) < \infty$, Condition C1 holds and a^* is finite. Then $r(G, \tilde{\delta}_n) - r(G, \delta_G) = O(\exp(-\tau n))$ for some positive constant $\tau = \tau(G)$ depending on the prior distribution G.

We provide the proof of Theorem 3.1 as follows.

Proof of Theorem 3.1

Let $S_1 = \{x | b^* + 1 \leq x \leq a^* - 1,\ H(x) < 0\}$ and $S_2 = \{x | b^* + 1 \leq x \leq a^* - 1,\ H(x) > 0\}$.. From (1.5), (1.6), (2.10), (2.11) and by the definition of a^* and b^*,

$$\begin{aligned} r(G, \tilde{\delta}_n) - r(G, \delta_G) = & \sum_{x=0}^{b^*} H(x)\ f(x)\ P\{\tilde{H}_n(x) < 0,\ H(x) > 0\} \\ & + \sum_{x\ \in\ S_1} [-H(x)]\ f(x)\ P\{\tilde{H}_n(x) \geq 0,\ H(x) < 0\} \\ & + \sum_{x\ \in\ S_2} H(x)\ f(x)\ P\{\tilde{H}_n(x) < 0,\ H(x) > 0\} \qquad (3.1) \\ & + \sum_{x=a^*+1}^{\infty} H(x)\ f(x)\ P\{\tilde{H}_n(x) < 0, H(x) > 0\} \\ \leq & \sum_{x=0}^{\infty} |H(x)| f(x), \end{aligned}$$

where $\sum_{x=c}^{d} \equiv 0$ if $d < c$ and $\sum_{x=\infty}^{\infty} \equiv 0$.

Since

$$\begin{aligned} |H(x)| &= |E[\Theta^2 | X = x]\ -\ 2\theta_0\ E[\Theta | X = x] + \theta_0^2 - c^2| \\ &\leq E[\Theta^2 | X\ =\ x]\ +\ 2\theta_0\ E[\Theta | X = x]\ +\ |\theta_0^2 - c^2|, \end{aligned}$$

$$\begin{aligned} \sum_{x=0}^{\infty} |H(x)|\ f(x)\ &= E_X[|H(X)|] \\ &\leq E_X[E[\Theta^2 | X]] +\ 2\theta_0\ E_X\ [E[\Theta | X]]\ +\ |\theta_0^2 - c^2| \qquad (3.2) \\ &= E[\Theta^2]\ +\ 2\theta_0\ E[\Theta]\ +\ |\theta_0^2 - c^2| \\ &< \infty \end{aligned}$$

by the assumption that $\int\ \theta^2\ dG(\theta)\ <\ \infty$.

Hence, to show that $r(G, \tilde{\delta}_n) - r(G, \delta_G) = o(1)$, it suffices to prove that for each x with $H(x) > 0$, $P\{\tilde{H}_n(x) < 0,\ H(x) > 0\} \to 0$ as $n \to \infty$, and for each x with $H(x) < 0$, $P\{\tilde{H}_n(x) \geq 0,\ H(x) < 0\} \to 0$ as $n \to \infty$.

Since $\tilde{H}_n(x) = \tilde{\varphi}_n(x+1)\ \tilde{\varphi}_n(x) - 2\theta_0\ \tilde{\varphi}_n(x) + \theta_0^2 - c^2$, where $\tilde{\varphi}_n(x) \to \varphi(x)$, $\tilde{\varphi}_n(x+1) \to \varphi(x+1)$ in probability by Lemma 2.1, so, $\tilde{H}_n(x) \to H(x)$ in probability. Hence, both $P\{\tilde{H}_n(x) \geq 0,\ H(x) < 0\}$ and $P\{\tilde{H}_n(x) < 0,\ H(x) > 0\}$ converge to 0 as $n \to \infty$. Therefore the proof is complete. $\square$

3.1 Preliminary Analysis

For providing a precise proof for Theorem 3.2, we first do certain preliminary analysis. When $b^* = -1$, in (3.1), the term $\sum_{x=0}^{b^*} H(x)\ f(x)\ P\{\tilde{H}_n(x) < 0,\ H(x) > 0\}$ is equal to 0. Hence, in the following, we may assume that $b^* \geq 0$.

Define the following events:

$$E_1 = \{\tilde{b}_n = b^*,\ \tilde{a}_n = a^*\},$$

$$A_1 = \{\tilde{b}_n \leq b^* - 1 \text{ or } \tilde{b}_n > b^* + 1\},$$

$$A_2 = \{\tilde{b}_n = b^* + 1\},$$

$$B_1 = \{\tilde{a}_n \leq a^* - 1 \text{ or } \tilde{a}_n > a^* + 1\},$$

$$B_2 = \{\tilde{a}_n = a^* + 1\},$$

$$E_2 = \{m_n \leq b^* \text{ and } M_n \geq a^* + 2\}.$$

Then, $E_1^c = \bigsqcup_{i=1}^{2} \bigsqcup_{j=1}^{2} (A_i\ B_j)$ and $E_2^c = \{m_n > b^* \text{ or } M_n < a^* + 2\}$.

Conditioning on $\underset{\sim}{X}(n)$, the regret Bayes risk of $\tilde{\delta}_n$ is:

$$r(G, \tilde{\delta}_n | \underset{\sim}{X}(n)) - r(G, \delta_G)$$

$$= \sum_{x=0}^{\infty} f(x)\ H(x)[\tilde{\delta}_n(x) - \delta_G(x)] I_{E_2^c}$$

$$+ \sum_{x=0}^{\infty} f(x)\ H(x)\ [\tilde{\delta}_n(x) - \delta_G(x)] I_{E_1 E_2}$$

$$+ \sum_{x=0}^{\infty} f(x)\ H(x)\ [\tilde{\delta}_n(x) - \delta_G(x)] \sum_{i=1}^{2} \sum_{j=1}^{2} I_{A_i B_j E_2}. \tag{3.3}$$

By the definitions of the events E_1 and $A_2 B_2$ and the tests $\tilde{\delta}_n$ and δ_G,

$$\sum_{x=0}^{\infty} f(x)\ H(x)\ [\tilde{\delta}_n(x) - \delta_G(x)]\ I_{E_1 E_2}$$

$$= \sum_{x=b^*+1}^{a^*-1} f(x)\ H(x)\ [\tilde{\delta}_n(x) - \delta_G(x)]\ I_{E_1 E_2}; \tag{3.4}$$

$$\sum_{x=0}^{\infty} f(x)\ H(x)\ [\tilde{\delta}_n(x) - \delta_G(x)] I_{A_2 B_2 E_2}$$

$$= \sum_{x=b^*+1}^{a^*} f(x)\ H(x)\ [\tilde{\delta}_n(x) - \delta_G(x)]\ I_{A_2 B_2 E_2}. \tag{3.5}$$

Therefore, the overall Bayes risk of the test $\tilde{\delta}_n$ is:

$$r(G, \tilde{\delta}_n) - r(G, \delta_G)$$

$$= E_{\underset{\sim}{X}(n)} [r(G, \tilde{\delta}_n | \underset{\sim}{X}(n)) - r(G, \delta_G)]$$

$$= \sum_{x=0}^{\infty} f(x)\ H(x)\ E_{\underset{\sim}{X}(n)}[(\tilde{\delta}_n(x) - \delta_G(x)) I_{E_2^c}]$$

$$+ \sum_{x=b^*+1}^{a^*-1} f(x)\ H(x)\ E_{\underset{\sim}{X}(n)}[(\tilde{\delta}_n(x) - \delta_G(x)) I_{E_1 E_2}]$$

$$+ \sum_{x=b^*+1}^{a^*} f(x)\ H(x)\ E_{\underset{\sim}{X}(n)}[(\tilde{\delta}_n(x) - \delta_G(x)) I_{A_2 B_2 E_2}]$$

$$+ \sum_{x=0}^{\infty} f(x)\ H(x)\ E_{\underset{\sim}{X}(n)}[(\tilde{\delta}_n(x) - \delta_G(x))(I_{A_1B_1E_2}\ +\ I_{A_1B_2E_2}\ +\ I_{A_2B_1E_2})]. \tag{3.6}$$

We can obtain the following results:

(a)

$$\sum_{x=0}^{\infty} f(x)\ H(x)\ E_{\underset{\sim}{X}(n)}[(\tilde{\delta}_n(x) - \delta_G(x))\ I_{E^c}]$$

$$= E_{(\underset{\sim}{X}(n),X)}\ [H(X)(\tilde{\delta}_n(X) - \delta_G(X))\ I_{E_2^c}]$$

$$\leq E_{(\underset{\sim}{X}(n),X)}\ [|H(X)|I_{E_2^c}]$$

$$= E_X\ [|H(X)|]E_{\underset{\sim}{X}(n)}[I_{E_2^c}]$$

since X and $\underset{\sim}{X}(n)$ are independent.

From (3.2), $E_X[|H(X)|] < \infty$. By the definitions of m_n and M_n,

$$E_{\underset{\sim}{X}(n)}[I_{E_2^c}] = P\{m_n > b^* \text{ or } M_n < a^* + 2\}$$

$$\leq P\{m_n > b^*\} + P\{M_n < a^* + 2\}$$

$$= [\overline{F}(b^*)]^n\ +\ [F(a^* + 2)]^n$$

$$=\ \exp[-n\ \ell n(\overline{F}(b^*))^{-1}] +\ \exp[-n\ \ell n(F(a^* + 2))^{-1}]$$

$$= O(\exp(-\tau_1 n)),$$

where F is the marginal distribution function of the random variable X, $\overline{F}(x) = 1 - F(x)$ and $\tau_1 =\ \min(\ell n(\overline{F}(b^*))^{-1}, \ell n(F(a^* + 2))^{-1}) > 0$. Therefore,

$$\sum_{x=0}^{\infty}\ f(x)\ H(x)E_{\underset{\sim}{X}(n)}[(\tilde{\delta}_n(x) - \delta_G(x))I_{E_2^c}] = O(\exp(-\tau_1 n)). \tag{3.7}$$

(b)

$$\sum_{x=b^*+1}^{a^*-1} f(x)\ H(x)\ E_{\underset{\sim}{X}(n)}[(\tilde{\delta}_n(x)-\delta_G(x))I_{E_1E_2}]$$

$$= \sum_{x\ \in\ S_1} [-H(x)]f(x)P\{\tilde{\delta}_n(x)=0, \delta_G(x)=1 \text{ and } E_1E_2 \text{ occurs}\} \tag{3.8}$$

$$+ \sum_{x\ \in\ S_2} H(x)\ f(x)P\{\tilde{\delta}_n(x)=1,\ \delta_G(x)=0 \text{ and } E_1E_2 \text{ occurs}\}.$$

(c)

$$\sum_{x=b^*+1}^{a^*} H(x)\ f(x)E_{\underset{\sim}{X}(n)}[(\tilde{\delta}_n(x)-\delta_G(x))I_{A_2B_2E_2}]$$

$$\begin{aligned} &= \sum_{x\ \in\ S_1} [-H(x)]\ f(x)P\{\tilde{\delta}_n(x)=0, \delta_G(x)=1 \text{ and } A_2B_2E_2 \text{ occurs}\} \\ &+ \sum_{x\ \in\ S_2} H(x)\ f(x)\ P\{\tilde{\delta}_n(x)=1, \delta_G(x)=0, \text{ and } A_2B_2E_2 \text{ occurs}\} \\ &+ H(a^*)\ f(a^*)P\{\tilde{\delta}_n(a^*)=1, \delta_G(a^*)=0 \text{ and } A_2B_2E_2 \text{ occurs}\}. \end{aligned} \tag{3.9}$$

(d)

$$\sum_{x=0}^{\infty} f(x)\ H(x)\ E_{\underset{\sim}{X}(n)}[(\tilde{\delta}_n(x)-\delta_G(x))(I_{A_1B_1E_2}+I_{A_1B_2E_2}+I_{A_2B_1E_2})]$$

$$\begin{aligned} &\le E_{(\underset{\sim}{X}(n),X)}[H(X)(\tilde{\delta}_n(X)-\delta_G(X))(I_{A_1E_2}+I_{B_1E_2})] \\ &\le E_{(\underset{\sim}{X}(n),X)}[|H(X)|(I_{A_1E_2}+I_{B_1E_2})] \\ &= E_X[|H(X)|]\{E_{\underset{\sim}{X}(n)}[I_{A_1E_2}]+E_{\underset{\sim}{X}(n)}[I_{B_1E_2}]\}. \end{aligned} \tag{3.10}$$

In (3.8) - (3.10), $0 \le \sum_{x\ \in\ S_1} [-H(x)]f(x) \le E_X[|H(X)|] < \infty$, $0 \le \sum_{x\ \in\ S_2} H(x)\ f(x) \le E_X[|H(X)|] < \infty$. Note that under the assumption that a^* is finite, both S_1 and S_2 are finite sets. Therefore, to investigate the asymptotic behavior of the regret Bayes risk

$r(G, \tilde{\delta}_n) - r(G, \delta_G)$, it suffices to investigate the asymptotic behaviors of the following terms:

$$D_1 = P\{\tilde{\delta}_n(x) = 0, \delta_G(x) = 1 \text{ and } E_1E_2 \text{ occurs }\} \text{ for } x \in S_1,$$

$$D_2 = P\{\tilde{\delta}_n(x) = 1, \delta_G(x) = 0, \text{ and } E_1E_2 \text{ occurs }\} \text{ for } x \in S_2,$$

$$D_3 = P\{\tilde{\delta}_n(x) = 0, \delta_G(x) = 1, \text{ and } A_2B_2E_2 \text{ occurs }\} \text{ for } x \in S_1,$$

$$D_4 = P\{\tilde{\delta}_n(x) = 1, \delta_G(x) = 0 \text{ and } A_2B_2E_2 \text{ occurs }\} \text{ for } x \in S_2,$$

$$D_5 = P\{\tilde{\delta}_n(a^*) = 1, \delta_G(a^*) = 0 \text{ and } A_2B_2E_2 \text{ occurs }\}$$

$$D_6 = E_{\underset{\sim}{X}(n)}[I_{A_1E_2}] \text{ and } D_7 = E_{\underset{\sim}{X}(n)}[I_{B_1E_2}].$$

(I) By the definitions of the tests $\tilde{\delta}_n$ and δ_G, for $x \in S_1$,

$$\begin{aligned}
D_1 &= P\{\tilde{\delta}_n(x) = 0, \delta_G(x) = 1 \text{ and } E_1E_2 \text{ occurs }\} \\
&= P\{\tilde{H}_n(x) - H(x) > -H(x) \text{ and } E_1E_2 \text{ occurs }\} \\
&= P\{[\tilde{\varphi}_n(x+1) - \varphi(x+1)]\tilde{\varphi}_n(x) + [\varphi(x+1) - 2\theta_0][\tilde{\varphi}_n(x) - \varphi(x)] > -H(x) \\
&\qquad \text{and } E_1E_2 \text{ occurs }\} \\
&\le P\{[\tilde{\varphi}_n(x+1) - \varphi(x+1)]\tilde{\varphi}_n(x) > -\frac{H(x)}{2} \text{ and } E_1E_2 \text{ occurs }\} \\
&\quad + P\{[\varphi(x+1) - 2\theta_0][\tilde{\varphi}_n(x) - \varphi(x)] > -\frac{H(x)}{2} \text{ and } E_1E_2 \text{ occurs }\} \\
&\equiv D_{11} + D_{12}.
\end{aligned}$$

On E_1, for $x \in S_1$, $0 < \theta_0 - c \le \tilde{\varphi}_n(x) < \theta_0 + c < 2\theta_0$. So

$$D_{11} \le P\{\tilde{\varphi}_n(x+1) - \varphi(x+1) > -\frac{H(x)}{4\theta_0} \text{ and } E_1E_2 \text{ occurs}\}.$$

Also, for $x \in S_1$, if $x+1 < a^*$, then by the definition of a^*, $\theta_0 - c \le \varphi(x+1) < \theta_0 + c < 2\theta_0$.

Hence,

$$D_{12} = P\{\tilde{\varphi}_n(x) - \varphi(x) < \frac{H(x)}{2[2\theta_0 - \varphi(x+1)]} \text{ and } E_1E_2 \text{ occurs } \}$$

$$\leq P\{\tilde{\varphi}_n(x) - \varphi(x) < \frac{H(x)}{4\theta_0} \text{ and } E_1E_2 \text{ occurs } \}.$$

When $x + 1 = a^*$, there are three cases: $\varphi(a^*) < 2\theta_0$ or $\varphi(a^*) = 2\theta_0$ or $\varphi(a^*) > 2\theta_0$. Hence,

$$D_{12} = P\{[\varphi(a^*) - 2\theta_0][\tilde{\varphi}_n(x) - \varphi(x)] > -\frac{H(x)}{2} \text{ and } E_1E_2 \text{ occurs } \}$$

$$\leq \begin{cases} P\{\tilde{\varphi}_n(x) - \varphi(x) < \frac{H(x)}{4\theta_0} \text{ and } E_1E_2 \text{ occurs } \} & \text{if } \varphi(a^*) < 2\theta_0, \\ 0 & \text{if } \varphi(a^*) = 2\theta_0, \\ P\{\tilde{\varphi}_n(x) - \varphi(x) > \frac{-H(x)}{2\varphi(a^*)} \text{ and } E_1E_2 \text{ occurs } \} & \text{if } \varphi(a^*) > 2\theta_0. \end{cases}$$

(II) For $x \in S_2$,

$$D_2 = P\{\tilde{\delta}_n(x) = 1, \delta_G(x) = 0 \text{ and } E_1E_2 \text{ occurs } \}$$

$$= P\{\tilde{H}_n(x) - H(x) < -H(x) \text{ and } E_1E_2 \text{ occurs } \}$$

$$= P\{[\tilde{\varphi}_n(x+1) - \varphi(x+1)]\tilde{\varphi}_n(x) + [\varphi(x+1) - 2\theta_0][\tilde{\varphi}_n(x) - \varphi(x)] < -H(x) \text{ and } E_1E_2 \text{ occurs } \}$$

$$\leq P\{[\tilde{\varphi}_n(x+1) - \varphi(x+1)]\tilde{\varphi}_n(x) < -\frac{H(x)}{2} \text{ and } E_1E_2 \text{ occurs } \}$$

$$+ P\{[\varphi(x+1) - 2\theta_0][\tilde{\varphi}_n(x) - \varphi(x)] < -\frac{H(x)}{2} \text{ and } E_1E_2 \text{ occurs } \}$$

$$\equiv D_{21} + D_{22},$$

where by an argument analogous to that for D_1,

$$D_{21} \leq P\{\tilde{\varphi}_n(x+1) - \varphi(x+1) < -\frac{H(x)}{4\theta_0} \text{ and } E_1E_2 \text{ occurs } \},$$

and

$$D_{22} = P\{[\varphi(x+1) - 2\theta_0][\tilde{\varphi}_n(x) - \varphi(x)] < -\frac{H(x)}{2} \text{ and } E_1E_2 \text{ occurs}\}$$

$$\leq \begin{cases} P\{\tilde{\varphi}_n(x) - \varphi(x) > \frac{H(x)}{4\theta_0} \text{ and } E_1E_2 \text{ occurs}\} & \text{if } \varphi(x+1) < 2\theta_0, \\ 0 & \text{if } \varphi(x+1) = 2\theta_0 \\ P\{\tilde{\varphi}_n(x) - \varphi(x) < -\frac{H(x)}{2\varphi(a^*)} \text{ and } E_1E_2 \text{ occurs}\} & \text{when } x+1 = a^* \text{ and } \varphi(a^*) > 2\theta_0. \end{cases}$$

(III) Similarly, on $A_2B_2E_2$, for $x \in S_1$, $0 < \tilde{\varphi}_n(x) < \theta_0 + c < 2\theta_0$, and

$$D_3 = P\{\tilde{\delta}_n(x) = 0, \delta_G(x) = 1 \text{ and } A_2B_2E_2 \text{ occurs}\}$$

$$\leq P\{\tilde{\varphi}_n(x+1) - \varphi(x+1) > -\frac{H(x)}{4\theta_0} \text{ and } A_2B_2E_2 \text{ occurs}\}$$

$$+ \begin{cases} P\{\tilde{\varphi}_n(x) - \varphi(x) < \frac{H(x)}{4\theta_0} \text{ and } A_2B_2E_2 \text{ occurs}\} & \text{if } \varphi(x+1) < 2\theta_0 \\ 0 & \text{if } \varphi(x+1) = 2\theta_0, \\ P\{\tilde{\varphi}_n(x) - \varphi(x) > \frac{-H(x)}{2\varphi(a^*)} \text{ and } A_2B_2E_2 \text{ occurs}\} & \text{when } x+1 = a^* \text{ and } \varphi(a^*) > 2\theta_0. \end{cases}$$

(IV) On $A_2B_2E_2$, for $x \in S_2$, $0 < \tilde{\varphi}_n(x) < \theta_0 + c < 2\theta_0$,

$$D_4 = P\{\tilde{\delta}_n(x) = 1, \delta_G(x) = 0 \text{ and } A_2B_2E_2 \text{ occurs}\}$$

$$\leq P\{\tilde{\varphi}_n(x+1) - \varphi(x+1) < -\frac{H(x)}{4\theta_0} \text{ and } A_2B_2E_2 \text{ occurs}\}$$

$$+ \begin{cases} P\{\tilde{\varphi}_n(x) - \varphi(x) > \frac{H(x)}{4\theta_0} \text{ and } A_2B_2E_2 \text{ occurs}\} & \text{if } \varphi(x+1) < 2\theta_0, \\ 0 & \text{if } \varphi(x+1) = 2\theta_0, \\ P\{\tilde{\varphi}_n(x) - \varphi(x) < -\frac{H(x)}{2\varphi(a^*)}, \text{ and } A_2B_2E_2 \text{ occurs }\} & \text{if } x+1 = a^* \text{ and } \varphi(a^*) > 2\theta_0. \end{cases}$$

(V)

$$
\begin{aligned}
D_5 &= P\{\tilde{\delta}_n(a^*) = 1, \delta_G(a^*) = 0 \text{ and } A_2B_2E_2 \text{ occurs }\} \\
&\le P\{\tilde{\varphi}_n(a^*+1) - \varphi(a^*+1) < -\frac{H(a^*)}{4\theta_0} \text{ and } A_2B_2E_2 \text{ occurs }\} \\
&+ \begin{cases} P\{\tilde{\varphi}_n(a^*) - \varphi(a^*) > \frac{H(a^*)}{4\theta_0} \text{ and } A_2B_2E_2 \text{ occurs }\} & \text{if } \varphi(a^*+1) < 2\theta_0, \\ 0 & \text{if } \varphi(a^*+1) = 2\theta_0, \\ P\{\tilde{\varphi}_n(a^*) - \varphi(a^*) < -\frac{H(a^*)}{2\varphi(a^*+1)} \text{ and } A_2B_2E_2 \text{ occurs }\} & \text{if } \varphi(a^*+1) > 2\theta_0. \end{cases}
\end{aligned}
$$

(VI)

$$
\begin{aligned}
D_6 &= E_{\underset{\sim}{X}(n)}[I_{A_1E_2}] \\
&= P\{(\tilde{b}_n \le b^* - 1 \text{ or } \tilde{b}_n > b^* + 1) \text{ and } E_2 \text{ occurs }\} \\
&= P\{\tilde{b}_n \le b^* - 1 \text{ and } E_2 \text{ occurs }\} \\
&\quad + P\{\tilde{b}_n > b^* + 1 \text{ and } E_2 \text{ occurs }\} \\
&= P\{\tilde{\varphi}_n(b^*) \ge \theta_0 - c \text{ and } E_2 \text{ occurs }\} \\
&\quad + P\{\tilde{\varphi}_n(b^*+2) < \theta_0 - c \text{ and } E_2 \text{ occurs }\} \\
&= P\{\tilde{\varphi}_n(b^*) - \varphi(b^*) \ge (\theta_0 - c) - \varphi(b^*) \text{ and } E_2 \text{ occurs }\} \\
&\quad + P\{\tilde{\varphi}_n(b^*+2) - \varphi(b^*+2) < \theta_0 - c - \varphi(b^*+2) \text{ and } E_2 \text{ occurs }\}.
\end{aligned}
$$

Here, note that, by the definition of b^*, $(\theta_0 - c) - \varphi(b^*) > 0$. Also, since the prior distribution

G is non-degenerate, by the definition of b^* again, $(\theta_0 - c) - \varphi(b^* + 2) < 0$.

$$
\begin{aligned}
\text{(VII)}\quad D_7 &= E_{\underset{\sim}{X}(n)}[I_{B_1 E_2}] \\
&= P\{(\tilde{a}_n \le a^* - 1 \text{ or } \tilde{a}_n > a^* + 1) \text{ and } E_2 \text{ occurs } \} \\
&= P\{\tilde{a}_n \le a^* - 1 \text{ and } E_2 \text{ occurs } \} \\
&\quad + \quad P\{\tilde{a}_n > a^* + 1 \text{ and } E_2 \text{ occurs } \} \\
&= P\{\tilde{\varphi}_n(a^* - 1) \ge \theta_0 + c \text{ and } E_2 \text{ occurs } \} \\
&\quad + \quad P\{\tilde{\varphi}_n(a^* + 1) < \theta_0 + c \text{ and } E_2 \text{ occurs } \} \\
&= P\{\tilde{\varphi}_n(a^* - 1) - \varphi(a^* - 1) \ge (\theta_0 + c) - \varphi(a^* - 1) \text{ and } E_2 \text{ occurs } \} \\
&\quad + \quad P\{\tilde{\varphi}_n(a^* + 1) - \varphi(a^* + 1) < (\theta_0 + c) - \varphi(a^* + 1) \text{ and } E_2 \text{ occurs } \}.
\end{aligned}
$$

Note that, here, $(\theta_0 + c) - \varphi(a^* - 1) > 0$ while $(\theta_0 + c) - \varphi(a^* + 1) < 0$.

Based on the above analysis for D_1 through D_7, to study the asymptotic behavior of the regret Bayes risk $r(G, \tilde{\delta}_n) - r(G, \delta_G)$, it suffices to investigate the asymptotic behavior of the following: $A_{1n}(x) = P\{\tilde{\varphi}_n(x) - \varphi(x) > d(x) \text{ and } E_2 \text{ occurs } \}$ for $b^* \le x \le a^*$, and $A_{2n}(x) = P\{\tilde{\varphi}_n(x) - \varphi(x) < -d(x) \text{ and } E_2 \text{ occurs } \}$ for $b^* + 1 \le x \le a^* + 1$, where $\varphi(x) > d(x) > 0, b^* \le x \le a^* + 1, d(x)$ is a suitably defined positive function and its definition is implicitly contained in the form of upper bounds of $D_i, i = 1 \ldots 7$, given previously.

3.2 Lemmas

In this subsection, we introduce certain lemmas which are helpful to investigate the asymptotic behaviors of $A_{1n}(x)$ and $A_{2n}(x)$.

The following lemma is from Liang (1991).

Lemma 3.1 Let $\{a_\ell\}_{\ell=1}^{\infty}$ be a sequence of real numbers and $\{b_\ell\}_{\ell=1}^{\infty}$ be a sequence of

nonincreasing positive numbers with $b_1 \le 1$. Then, for any positive constant d,

$$\sup_{n\ge1} \left|\sum_{\ell=1}^{n} a_\ell \, b_\ell\right| \ge (>) d \Rightarrow \sup_{n\ge1} \left|\sum_{\ell=1}^{n} a_\ell\right| \ge (>) d.$$

Let F_n be the empirical distribution based on $\underset{\sim}{X}(n) = (X_1, \ldots, X_n)$ and F the marginal distribution function of the random variable X.

Corollary 3.1 Let d be a fixed positive value and ℓ be a fixed nonnegative integer. Then under Condition C1, for n being sufficiently large, the following holds.

(a) $\sup\limits_{y\ge\ell} |[K_n(y) - K(y)] - [K_n(\ell-1) - K(\ell-1)]| \ge (>) d$

$\Rightarrow \sup\limits_{y\ge0} |F_n(y) - F(y)| \ge (>) \frac{d}{3}$.

(b) $\sup\limits_{y\ge\ell} |[\psi_n(y) - \psi(y)] - [\psi_n(\ell-1) - \psi(\ell-1)]| \ge (>) d$

$\Rightarrow \sup\limits_{y\ge0} |F_n(y) - F(y)| \ge (>) \frac{d}{3}$.

Proof: Since $\epsilon_n = o(n^{-1})$, under Condition C1, there exists an integer $N \equiv N(d)$ such that for $n \ge N, \epsilon_n \sum\limits_{x=0}^{\infty} w(x) < \frac{d}{3}$.

Next, note that

$$K_n(y) = \sum_{x=0}^{y} h_n(x) \, w(x)$$

$$= \sum_{x=0}^{y} f_n(x) \, \frac{w(x)}{a(x)} + \epsilon_n \sum_{x=0}^{y} w(x)$$

$$K(y) = \sum_{x=0}^{y} f(x) \, \frac{w(x)}{a(x)}.$$

Hence, for n being large enough such that $\epsilon_n \sum\limits_{x=0}^{\infty} w(x) < \frac{d}{3}$,

$$\sup_{y\ge\ell} \left|[K_n(y) - K(y)] - [K_n(\ell-1) - K(\ell-1)]\right| \ge (>) d$$

$$\Rightarrow \sup_{y\ge\ell} \left|\sum_{x=\ell}^{y} [f_n(x) - f(x)] \frac{w(x)}{a(x)} + \epsilon_n \sum_{x=\ell}^{y} w(x)\right| \ge (>) d$$

$$\Rightarrow \sup_{y\geq \ell} \mid \sum_{x=\ell}^{y} [f_n(x) - f(x)]\frac{w(x)}{a(x)} \mid \geq (>)\frac{2}{3}\, d$$

$$\Rightarrow \sup_{y\geq \ell} \mid \sum_{x=\ell}^{y} [f_n(x) - f(x)]\big| \geq (>)\frac{2}{3}d, \text{ by Lemma 3.1 under Condition C1,}$$

$$\Rightarrow \sup_{y\geq \ell} \big|[F_n(y) - F(y)] - [F_n(\ell-1) - F(\ell-1)]\big| \geq (>)\frac{2}{3}d$$

$$\Rightarrow \sup_{y\geq 0} \big|F_n(y) - F(y)\big| \geq (>)\ \frac{d}{3}.$$

Part (b) can be proved in a similar way. □

Define $d_1(x) = \varphi(x) + d(x)$, $d_2(x) = \varphi(x) - d(x)$. Note that $d_i(x) > 0$, $i = 1, 2$. For each $y \geq x$, define,

$$q_x(y) = d_2(x)\ \sum_{t=x}^{y}\ h(t)w(t) - \sum_{t=x}^{y}\ h(t+1)w(t).$$

Also, for each $z \leq x$, define

$$p_x(z) = -\sum_{t=z}^{x}\ h(t+1)w(t) + d_1(x)\ \sum_{t=z}^{x} h(t)w(t).$$

Lemma 3.2 (a) $q_x(y)$ is decreasing in y for $y \geq x$. Hence,

$$\begin{aligned} \max_{y\geq x}\ q_x(y) = q_x(x) &= h(x)w(x)[d_2(x) - \varphi(x)] \\ &= h(x)w(x)(-d(x)) < 0. \end{aligned}$$

(b) $p_x(z)$ is decreasing in z for $z \leq x$. Hence,

$$\begin{aligned} \min_{0\leq z\leq x}\ p_x(z) = p_x(x) = &\ w(x)h(x)[d_1(x) - \varphi(x)] \\ = &\ w(x)h(x)d(x) > 0. \end{aligned}$$

Proof: (a) For $y \geq x$, consider

$$\begin{aligned} q_x(y+1) - q_x(y) = & \ d_2(x)h(y+1)w(y+1) - h(y+2)w(y+1) \\ = & \ h(y+1)w(y+1)[d_2(x) - \frac{h(y+2)}{h(y+1)}] \\ = & \ h(y+1)w(y+1)[\varphi(x) - d(x) - \varphi(y+1)] \\ & <0 \end{aligned}$$

since $\varphi(x) - \varphi(y+1) < 0$ for $y \geq x$ and $d(x) > 0$. Hence, $q_x(y)$ is decreasing in y for $y \geq x$.
(b) For $z \leq x$, Consider

$$\begin{aligned} p_x(z) - p_x(z-1) = & \ h(z)w(z-1) - d_1(x)h(z-1)w(z-1) \\ = & \ h(z-1)w(z-1)\left[\frac{h(z)}{h(z-1)} - d_1(x)\right] \\ = & \ h(z-1)w(z-1)[\varphi(z-1) - \varphi(x) - d(x)] \\ & <0 \end{aligned}$$

since $\varphi(z-1) - \varphi(x) < 0$ for $0 \leq z \leq x$ and $d(x) > 0$. So, $p_x(z)$ is decreasing in z for $0 \leq z \leq x$. $\square$

Let $\Delta_{\psi_n}(x) = \psi_n(x) - \psi(x), \Delta_{K_n}(x) = K_n(x) - K(x)$ and $\Delta_{F_n}(x) = F_n(x) - F(x)$.

Lemma 3.3 (a) For each $b^* \leq x \leq a^*$, and n being sufficiently large,

$$\{ \ \tilde{\varphi}_n(x) - \varphi(x) > d(x) \text{ and } E_2 \text{ occurs}\}$$

$$\subset \{ \ \sup_{y \geq 0} |\Delta_{F_n}(y)| > \frac{p_x(x)}{6} \min\big(1, \frac{1}{d_1(x)}\big)\}.$$

(b) For each $b^* + 1 \leq x \leq a^* + 1$, and n being sufficiently large,

$$\{ \ \tilde{\varphi}_n(x) - \varphi(x) < -d(x) \text{ and } E_2 \text{ occurs}\}$$

$$\subset \ \{ \ \sup_{y \geq 0} |\Delta_{F_n}(y)| > -\frac{q_x(x)}{6} \min\big(1, \frac{1}{d_2(x)}\big)\}.$$

Proof: (a) For $b^* \le x \le a^*$, by (2.20) and Corollary 3.1, for n being sufficiently large,

$$\{\tilde{\varphi}_n(x) - \varphi(x) > d(x) \text{ and } E_2 \text{ occurs}\}$$

$$= \{\tilde{\varphi}_n(x) > \varphi(x) + d(x) = d_1(x) \text{ and } E_2 \text{ occurs}\}$$

$$\subset \{[\psi_n(x) - \psi_n(y-1)] - [K_n(x) - K_n(y-1)]d_1(x) > 0 \text{ for some } x \ge y \ge m_n \text{ and } E_2 \text{ occurs}\}$$

$$\subset \bigsqcup_{y=0}^{x} \{[\Delta_{\psi_n}(x) - \Delta_{\psi_n}(y-1)] - d_1(x)[\Delta_{K_n}(x) - \Delta_{K_n}(y-1)] > p_x(y)\}$$

$$\subset \bigsqcup_{y=0}^{x} \{[\Delta_{\psi_n}(x) - \Delta_{\psi_n}(y-1)] - d_1(x)[\Delta_{K_n}(x) - \Delta_{K_n}(y-1)] > p_x(x)\}$$

$$\subset \bigsqcup_{y=0}^{x} \{[\Delta_{\psi_n}(x) - \Delta_{\psi_n}(y-1)] > \frac{p_x(x)}{2} \text{ or } [\Delta_{K_n}(x) - \Delta_{K_n}(y-1)] < -\frac{p_x(x)}{2d_1(x)}\}$$

$$\subset \{\sup_{y\ge 0} |\Delta_{F_n}(y)| > \frac{p_x(x)}{6} \text{ or } \sup_{y\ge 0} |\Delta_{F_n}(y)| > \frac{p_x(x)}{6d_1(x)}\}$$

$$= \{\sup_{y\ge 0} |\Delta_{F_n}(y)| > \frac{p_x(x)}{6} \min\left(1, \frac{1}{d_1(x)}\right)\}.$$

(b) For each $b^* + 1 \le x \le a^* + 1$, by (2.16) and Corollary 3.1, for n being large enough,

$$\{\tilde{\varphi}_n(x) - \varphi(x) < -d(x) \text{ and } E_2 \text{ occurs}\}$$

$$= \{\tilde{\varphi}_n < \varphi(x) - d(x) = d_2(x) \text{ and } E_2 \text{ occurs}\}$$

$$\subset \{[\psi_n(y) - \psi_n(x-1)] - d_2(x)[K_n(y) - K_n(x-1)] < 0 \text{ for some } x \le y \le M_n \text{ and } E_2 \text{ occurs}\}$$

$$= \{[\Delta_{\psi_n}(y) - \Delta_{\psi_n}(x-1)] - d_2(x)[\Delta_{K_n}(y) - \Delta_{K_n}(x-1)] < q_x(y) \text{ for some } x \le y \le M_n \text{ and } E_2 \text{ occurs}\}$$

$$\subset \{[\Delta_{\psi_n}(y) - \Delta_{\psi_n}(x-1)] - d_2(x)[\Delta_{K_n}(y) - \Delta_{K_n}(x-1)] < q_x(x) \text{ for some } x \le y\}$$

$$\subset \{\Delta_{\psi_n}(y) - \Delta_{\psi_n}(x-1) < \frac{q_x(x)}{2} \text{ or } [\Delta_{K_n}(y) - \Delta_{K_n}(x-1)] > -\frac{q_x(x)}{2d_2(x)} \text{ for some } x \le y\}$$

$$\subset \{\sup_{y\ge x} |\Delta_{\psi_n}(y) - \Delta_{\psi_n}(x-1)| > \frac{-q_x(x)}{2} \text{ or } \sup_{y\ge x} |\Delta_{K_n}(y) - \Delta_{K_n}(x-1)| > \frac{-q_x(x)}{2d_2(x)}\}$$

$$\subset \{\sup_{y\ge 0} |\Delta_{F_n}(y)| > -\frac{q_x(x)}{6}, \text{ or } \sup_{y\ge 0} |\Delta_{F_n}(y)| > -\frac{q_x(x)}{6d_2(x)}\}$$

$$= \{\sup_{y\ge 0} |\Delta_{F_n}(y)| \ge -\frac{q_x(x)}{6} \min\big(1, \frac{1}{d_2(x)}\big)\}. \qquad \square$$

3.3 Proof of Theorem 3.2

Let $\tau_2 = \min_{b^* \le x \le a^*} \frac{p_x(x)}{6} \min\big(1, \frac{1}{d_1(x)}\big)$, $\tau_3 = \min_{b^*+1 \le x \le a^*+1} \frac{-q_x(x)}{6} \min\big(1, \frac{1}{d_2(x)}\big)$. Note that both τ_2 and τ_3 are positive. Hence $\tau \equiv \min(\tau_1,\ \tau_2,\ \tau_3) > 0$.

From Lemma 3.3, for each $b^* \le x \le a^*$,

$$\begin{aligned} &P\{\tilde{\varphi}_n(x) - \varphi(x) > d(x) \text{ and } E_2 \text{ occurs}\} \\ &\le P\{\sup_{y\ge 0}\ |\Delta_{F_n}(y)| > \frac{p_x(x)}{6} \min\big(1, \frac{1}{d_1(x)}\big)\} \\ &\le P\{\sup_{y\ge 0} |\Delta_{F_n}(y)| > \tau_2\} \\ &\le d\ \ \exp(-2n\tau_2^2), \end{aligned} \tag{3.11}$$

by the exponential type inequality of Dvoretzky, Kiefer and Wolfowitz (1956), where d is a positive constant independent of the distribution function F.

Also, for $b^* + 1 \le x \le a^* + 1$, from Lemma 3.3,

$$\begin{aligned} &P\{\tilde{\varphi}_n(x) - \varphi(x) < -d(x) \text{ and } E_2 \text{ occurs}\} \\ &\le P\{\sup_{y\ge 0} |\Delta_{F_n}(y)| > \frac{-q_x(x)}{6} \min\big(1, \frac{1}{d_2(x)}\big)\} \\ &\le P\{\sup_{y\ge 0} |\Delta_{F_n}(y)| > \tau_3\} \\ &\le d\ \ \exp(-2n\tau_3^2). \end{aligned} \tag{3.12}$$

Now, combining the results of Section 3.1, (3.7), (3.11) and (3.12), we conclude that $r(G, \tilde{\delta}_n) - r(G, \delta_G) = O(\exp(-\tau n))$. $\square$

4. Example, Remarks and Further Results

4.1 Examples

We use the following two examples to illustrate the choice of the positive numbers $\{w(x)\}_{x=0}^{\infty}$ and demonstrate the asymptotic optimality of the empirical Bayes two-tail test $\tilde{\delta}_n$.

Example 1. (The Poisson distribution). Suppose that

$$f(x|\theta) = e^{-\theta}\theta^x/x!, \ x = 0, \ 1, \ 2, \ \ldots; \text{ and } 0 < \theta < \infty.$$

Then, $a(x) = \frac{1}{x!}$. Thus we let $w(x) = a(x+\ell) = \frac{1}{(x+\ell)!}$, where ℓ is a fixed, positive integer. Then, both $\frac{w(x)}{a(x)}$ and $\frac{w(x)}{a(x+1)}$ are nonincreasing in x, bounded above by 1 and $\sum_{x=0}^{\infty} w(x) < \infty$. Hence the requirements of Condition C1 are met.

Assume that the prior distribution G is a member of the family of gamma distribution with density function $g(\theta|k, \beta)$ of the form

$$g(\theta|k,\beta) = \frac{\beta^k}{\Gamma(k)} \theta^{k-1} e^{-\beta\theta}, \ 0 < \theta < \infty, \ k > 0, \beta > 0.$$

Then, $\varphi(x) = \frac{x+k}{1+\beta}$, which tends to ∞ as $x \to \infty$. Therefore, for finite $0 < c < \theta_0, a^* < \infty$. Then, by Theorem 3.2, we have $r(G, \tilde{\delta}_n) - r(G, \delta_G) = O(\exp(-\tau n))$ for some $\tau > 0$.

Example 2 (The Negative binomial distribution). Suppose that

$$f(x|\theta) = \binom{x+r-1}{r-1}\theta^x(1-\theta)^r, \ x = 0, 1, \ 2, \ \ldots; 0 < \theta < 1,$$

where r is a fixed, positive integer. Then $a(x) = \binom{x+r-1}{r-1}$. We let $w(x) = \frac{1}{(x+1)^2}$, so that both $\frac{w(x)}{a(x)}$ and $\frac{w(x)}{a(x+1)}$ are nonincreasing in x and bounded above by 1. Also $\sum_{x=0}^{\infty} w(x) < \infty$. Hence, the requirements of Condition C1 are met.

Suppose that the prior distribution G is a member of the family of beta distribution with parameter (α, β). Then, $\varphi(x) = \frac{x+\alpha}{x+\alpha+\beta+r}$, which tends to 1 as $x \to \infty$. Thus for

$0 < c < \theta_0 < 1$ such that $\theta_0 + c < 1$, $a^* < \infty$. Therefore, by Theorem 3.2, $r(G, \tilde{\delta}_n) - r(G, \delta_G) = O(\exp(-\tau n))$ for some $\tau > 0$.

Wei (1991) considered several situations about the behavior of the tail probability of the prior distribution G under which his proposed empirical Bayes test δ_n^W may achieve a rate near the best possible rate of convergence of order $O(n^{-1})$. We may also apply those conditions to the empirical Bayes test $\tilde{\delta}_n$. For example, under the assumption of Theorem 4.2 of Wei (1991), one can see that $\varphi(x) \to \infty$ as $x \to \infty$. Thus $H(x) > 0$ for x being sufficiently large and $a^* < \infty$. Therefore $r(G, \tilde{\delta}_n) - r(G, \delta_G) = O(\exp(-\tau n))$ for some $\tau > 0$. We see that under the same conditions, $\tilde{\delta}_n$ has a rate of convergence much faster than that of δ_n^W. Basically, Wei's approach is along the line of Johns and Van Ryzin (1971), in which one needs to treat the asymptotic behavior of each term in an infinite series of the regret Bayes risk. Our approach is somewhat different from Wei's. We mimick the behavior of the unknown Bayes two tail test δ_G, so that one only needs to deal with the asymptotic behavior of finite terms of probabilities, which have been discussed in details in Section 3.1.

If it is not possible or hard to find a sequence of positive numbers $\{w(x)\}_{x=0}^{\infty}$ to satisfy the requirements of Condition C1, we may consider the following alternative condition.

Condition C2. $\{w(x)\}_{x=0}^{\infty}$ is a sequence of positive numbers such that $\sum\limits_{x=0}^{\infty} \frac{w(x)}{a(x)} < \infty$, $\sum\limits_{x=0}^{\infty} \frac{w(x)}{a(x+1)} < \infty$ and $\sum\limits_{x=0}^{\infty} w(x) < \infty$.

It should be noted that a sequence of positive numbers $\{w(x)\}_{x=0}^{\infty}$ satisfying Condition C2 always exists. For example, we may let, for each $x = 0, 1, \ldots$, $w(x) = \frac{\min(a(x), a(x+1))}{\max(1, a(x), a(x+1))} b(x)$, where $\{b(x)\}_{x=0}^{\infty}$ is a sequence of positive numbers such that $\sum\limits_{x=0}^{\infty} b(x) < \infty$. Then it is straightforward to verify that $\{w(x)\}_{x=0}^{\infty}$ meets the requirements of Condition C2.

Based on the analysis given in the Subsection 3.1, to study the asymptotic optimality of the empirical Bayes test $\tilde{\delta}_n$ which is now constructed under Condition C2, one needs only to investigate the asymptotic behaviors of the following terms under Condition C2:

$$P\{\, \tilde{\varphi}_n(x) - \varphi(x) > d(x) \text{ and } E_2 \text{ occurs}\} \text{ for each } x, b^* \le x \le a^* \text{ and}$$

$$P\{\, \tilde{\varphi}_n(x) - \varphi(x) < -d(x) \text{ and } E_2 \text{ occurs}\} \text{ for each } x, b^* + 1 \le x \le a^* + 1.$$

Let $\beta = \frac{1}{6} \min_{b^* \leq x \leq a^*+1} \min(|q_x(x)|, p_x(x))$. Then, $\beta > 0$. Under Condition C2, there exists a positive integer $N \equiv N(\beta) > M$ such that for all $y \geq x \geq N$, the following hold,

$$\begin{cases} \varphi(a^*+1) \sum_{i=x}^{y} \frac{w(i)}{a(i)} < \beta, \\ \sum_{i=x}^{y} \frac{w(i)}{a(i+1)} < \beta, \\ [1+\varphi(a^*+1)] \sum_{i=x}^{y} w(i) < \beta. \end{cases} \tag{4.1}$$

Note that $|f_n(x) - f(x)| \leq 1$ for all x. Therefore, by (4.1), for $y \geq x \geq N$,

$$\begin{aligned} &|\Delta_{\psi_n}(y) - \Delta_{\psi_n}(x-1)| \\ &= |\sum_{i=x}^{y} [\frac{f_n(i+1)}{a(i+1)} + \epsilon_n - \frac{f(i+1)}{a(i+1)}]w(i)| \\ &\leq |\sum_{i=x}^{y} [f_n(i+1) - f(i+1)]\frac{w(i)}{a(i+1)}| + \epsilon_n \sum_{i=x}^{y} w(i) \qquad (4.2) \\ &\leq \sum_{i=x}^{y} \frac{w(i)}{a(i+1)} + \epsilon_n \sum_{i=x}^{y} w(i) \\ &< (1+ \in_n)\beta; \end{aligned}$$

$$\begin{aligned} &\varphi(a^*+1)|\Delta_{K_n}(y) - \Delta_{K_n}(x-1)| \\ &= \varphi(a^*+1)|\sum_{i=x}^{y} [\frac{f_n(i)}{a(i)} + \epsilon_n - \frac{f(i)}{a(i)}]w(i)| \\ &\leq \varphi(a^*+1)|\sum_{i=x}^{y} [f_n(i) - f(i)]\frac{w(i)}{a(i)}| + \varphi(a^*+1)\epsilon_n \sum_{i=x}^{y} w(i) \qquad (4.3) \\ &\leq \varphi(a^*+1)\sum_{i=x}^{y} \frac{w(i)}{a(i)} + \varphi(a^*+1)\epsilon_n \sum_{i=x}^{y} w(i) \\ &< (1+\epsilon_n)\beta. \end{aligned}$$

Note that since $\epsilon_n = o(n^{-1})$ for n sufficiently large, $2(1+\epsilon_n) < 3$.

Following the proof of Lemma 3.3, for each x with $b^* + 1 \le x \le a^* + 1$,

$$\begin{aligned}
&\{\tilde{\varphi}_n(x) - \varphi(x) < -d(x) \text{ and } E_2 \text{ occurs}\} \\
\subset &\bigsqcup_{y=x}^{\infty} \{[\Delta_{\psi_n}(y) - \Delta_{\psi_n}(x-1)] - d_2(x)[\Delta_{K_n}(y) - \Delta_{K_n}(x-1)] < q_x(x)\}
\end{aligned} \tag{4.4}$$

For each $y \ge N+1$,

$$\begin{aligned}
&[\Delta_{\psi_n}(y) - \Delta_{\psi_n}(x-1)] - d_2(x)[\Delta_{K_n}(y) - \Delta_{K_n}(x-1)] \\
= &[\Delta_{\psi_n}(y) - \Delta_{\psi_n}(N)] - d_2(x)[\Delta_{K_n}(y) - \Delta_{K_n}(N)] \\
&+ [\Delta_{\psi_n}(N) - \Delta_{\psi_n}(x-1)] - d_2(x)[\Delta_{K_n}(N) - \Delta_{K_n}(x-1)]
\end{aligned} \tag{4.5}$$

where $|\Delta_{\psi_n}(y) - \Delta_{\psi_n}(N)| < (1+\epsilon_n)\beta$ and $d_2(x)|\Delta_{K_n}(y) - \Delta_{K_n}(N)| < (1+\epsilon_n)\beta$, by (4.2) and (4.3), respectively, and noting that $0 < \varphi(x) \le \varphi(a^*+1)$ for $x \le a^*+1$. Hence, for each $y \ge N+1$, $b^*+1 \le x \le a^*+1$, for n being large enough so that $2(1+\epsilon_n) < 3$,

$$\begin{aligned}
&\{[\Delta_{\psi_n}(y) - \Delta_{\psi_n}(x-1)] - d_2(x)[\Delta_{K_n}(y) - \Delta_{K_n}(x-1)] < q_x(x)\} \\
\subset &\{[\Delta_{\psi_n}(N) - \Delta_{\psi_n}(x-1)] - d_2(x)[\Delta_{K_n}(N) - \Delta_{K_n}(x-1)] < \frac{q_x(x)}{2}\},
\end{aligned} \tag{4.6}$$

by noting that $q_x(x) < 0$. Also, for $x \le y \le N$

$$\begin{aligned}
&\{[\Delta_{\psi_n}(y) - \Delta_{\psi_n}(x-1)] - d_2(x)[\Delta_{K_n}(y) - \Delta_{K_n}(x-1)] < q_x(x)\} \\
\subset &\{[\Delta_{\psi_n}(y) - \Delta_{\psi_n}(x-1)] - d_2(x)[\Delta_{K_n}(y) - \Delta_{K_n}(x-1)] < \frac{q_x(x)}{2}\}.
\end{aligned} \tag{4.7}$$

Combining (4.4)-(4.7) yields, for n being sufficiently large, that

$$
\begin{aligned}
&\{\tilde{\varphi}_n(x) - \varphi(x) < -d(x) \text{ and } E_2 \text{ occurs}\} \\
\subset\ & \bigsqcup_{y=x}^{N} \{[\Delta_{\psi_n}(y) - \Delta_{\psi_n}(x-1)] - d_2(x)[\Delta_{K_n}(y) - \Delta_{K_n}(x-1)] < \frac{q_x(x)}{2}\} \\
=\ & \{\sum_{i=x}^{y} ([f_n(i+1) - d_2(x)f_n(i)] - [f(i+1) - d_2(x)f(i)])\frac{w(i)}{a(i)} \\
& \quad + \epsilon_n(1 - d_2(x)) \sum_{i=x}^{y} w(i) < \frac{q_x(x)}{2} \text{ for some } x \leq y \leq N\} \\
\subset\ & \{\sum_{i=x}^{y} ([f_n(i+1) - d_2(x)f_n(i)] - [f(i+1) - d_2(x)f(i)])\frac{w(i)}{a(i)} < \frac{q_x(x)}{4} \\
& \quad \text{for some } x \leq y \leq N\} \\
\subset\ & \{[f_n(i+1) - d_2(x)f_n(i)] - [f(i+1) - d_2(x)f(i)] < \frac{q_x(x)a(i)}{4w(i)(N-x+1)} \\
& \quad \text{for some } x \leq i \leq N\}.
\end{aligned} \tag{4.8}
$$

Therefore,

$$
\begin{aligned}
& P\{\tilde{\varphi}_n(x) - \varphi(x) < -d(x) \text{ and } E_2 \text{ occurs}\} \\
\leq\ & P\{[f_n(i+1) - d_2(x)f_n(i)] - [f(i+1) - d_2(x)f(i)] < \frac{q_x(x)a(i)}{4w(i)(N-x+1)} \\
& \quad \text{for some } x \leq i \leq N\} \\
\leq\ & \sum_{i=x}^{N} P\{[f_n(i+1) - d_2(x)f_n(i)] - [f(i+1) - d_2(x)f(i)] < \frac{q_x(x)a(i)}{4w(i)(N-x+1)}\} \\
\leq\ & \sum_{i=x}^{N} \exp\{-2n\left(\frac{q_x(x)a(i)}{4w(i)(N-x+1)}\right)^2 \times \frac{1}{(1+d_2(x))^2}\}.
\end{aligned} \tag{4.9}
$$

The last inequality in (4.9) is obtained by an application of Hoeffding's inequality and the following facts that $f_n(i+1) - d_2(x)f_n(i) = \frac{1}{n} \sum_{j=1}^{n} [I_{\{i+1\}}(X_j) - d_2(x)I_{\{i\}}(X_j)]$, where

$I_{\{i+1\}}(X_j) - d_2(x)I_{\{i\}}(X_j), j = 1, \ldots, n$, are iid, and $E[I_{\{i+1\}}(X_j) - d_2(x)I_{\{i\}}(X_j)] = f(i+1) - d_2(x)f(i)$, and $-d_2(x) \leq I_{\{i+1\}}(X_j) - d_2(x)I_{\{i\}}(X_j) \leq 1$.

Next, for each $x, b^* \leq x \leq a^*$, from the proof of Lemma 3.3a, for n being large enough,

$$\{\tilde{\varphi}_n(x) - \varphi(x) > d(x) \text{ and } E_2 \text{ occurs}\}$$

$$\subset \bigsqcup_{y=0}^{x} \{[\Delta_{\psi_n}(x) - \Delta_{\psi_n}(y-1)] - d_1(x)[\Delta_{K_n}(x) - \Delta_{K_n}(y-1)] > p_x(x)\}$$

$$= \{\sum_{i=y}^{x} \left([\frac{f_n(i+1)}{a(i)} + \epsilon_n - \frac{f(i+1)}{a(i)}]w(i) - d_1(x)[\frac{f_n(i)}{a(i)} + \epsilon_n - \frac{f(i)}{a(i)}]w(i) \right) > p_x(x)$$
$$\text{for some } 0 \leq y \leq x\}$$

$$\subset \{\sum_{i=y}^{x} \left([f_n(i+1) - d_1(x)f_n(i)] - [f(i+1) - d_1(x)f(i)] \right) \frac{w(i)}{a(i)} > \frac{p_x(x)}{2}$$
$$\text{for some } 0 \leq y \leq x\}$$

$$\subset \bigsqcup_{i=0}^{x} \{[f_n(i+1) - d_1(x)f_n(i)] - [f(i+1) - d_1(x)f(i)] > \frac{p_x(x)a(i)}{2w(i)(x+1)}\}. \tag{4.10}$$

Therefore, by Hoeffding's inequality,

$$P\{\tilde{\varphi}_n(x) - \varphi(x) > d(x) \text{ and } E_2 \text{ occurs}\}$$

$$\leq \sum_{i=0}^{x} P\{[f_n(i+1) - d_1(x)f_n(i)] - [f(i+1) - d_1(x)f(i)] > \frac{p_x(x)a(i)}{2w(i)(x+1)}\} \tag{4.11}$$

$$\leq \sum_{i=0}^{x} \exp\{-2n\left[\frac{p_x(x)a(i)}{2w(i)(x+1)}\right]^2 \frac{1}{[1+d_1(x)]^2}\}.$$

Let

$$\tau_4 = 2 \min_{b^*+1 \leq x \leq a^*+1} \min_{x \leq i \leq N} \left(\frac{q_x(x)a(i)}{4w(i)(N-x+1)(1+d_2(x))}\right)^2,$$

$$\tau_5 = 2 \min_{b^* \leq x \leq a^*} \min_{0 \leq i \leq x} \left(\frac{p_x(x)a(i)}{2w(i)(x+1)(1+d_1(x))}\right)^2.$$

Since N and a^* are finite numbers, $\tau_4 > 0$ and $\tau_5 > 0$. Now combining the results of Subsection 3.1, (4.8) and (4.10) and noting again that both N and a^* are finite, we can conclude the following theorem.

Theorem 4.1 Suppose that $\int \theta^2 dG(\theta) < \infty$, Condition C2 holds, and a^* is finite. Then, $r(G, \tilde{\delta}_n) - r(G, \delta_G) = O(\exp(-\tau_6 n))$ where $\tau_6 = \min(\tau_1, \tau_4, \tau_5) > 0$.

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